If it were a snake, it would have bitten you: Money in the New Keynesian Model*

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The New Keynesian literature focuses on rules-based interest rate policies, abstracting from the role of monetary aggregates. In the background, though, the *quantity equation* must hold — every transaction requires money, with money units used in multiple transactions within a period. What is often overlooked is that imposing a rules-based interest rate policy is equivalent to assuming a particular *money velocity* specification. Using this alternative specification, we derive the efficient money supply rule and show that determinate equilibria exist with money supply policy and a *fixed* nominal interest rate. We estimate a New Keynesian model with either conventional interest rate policy or our money market reinterpretation of the model, accounting for the policy rate lower bound (PRLB). The money market estimates exactly match the PRLB duration in the data, whereas the conventional estimates fall short by four years.

Keywords: New Keynesian model, Taylor rule, velocity of money, money demand, quantity equation, optimal monetary policy, Bayesian estimation, policy rate lower bound

JEL Codes: E40, E50, E52

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1 Introduction and the Textbook New Keynesian Model

The New Keynesian (NK) model in its current form has been well studied over the last 25 years, including the seminal works of Rotemberg and Woodford (1997) and Clarida et al. (2000). Textbook treatments of the model include Woodford (2003) and Galí (2015). This paper revisits the question: what is the role of the money market in the NK model? We show four things. First, picking a rules-based interest rate policy in the NK model is isomorphic to imposing a money demand structure with an exogenous money supply. Second, the determinacy condition on endogenous money supply policy generalizes the conventional condition in the NK model. Third, optimal money supply policy simultaneously targets inflation and the output gap, with the interest rate equal to the natural rate in equilibrium. Finally, estimating the NK model under our money market interpretation matches the binding policy rate lower bound (PRLB) constraint duration in the data, whereas conventional interest rate policy estimates do not.

This paper does not necessarily argue that the NK model is better understood by closing the model with a money demand specification rather than an interest rate policy rule. Instead, we show that (1) these two different specifications are indistinguishable under particular model parameterizations, and (2) this is not merely a theoretical curiosity. These two different interpretations of the model have important implications for monetary policy. This is true on the superficial level of thinking about the money supply or the interest rate as the primary tool of monetary policy, but that is not the only implication. We show that a determinate linear rational expectations equilibrium exists for reasonable parameterizations of a money growth rule *even with a permanently fixed nominal interest rate*. This implies that practical concerns about the effectiveness of monetary policy when the policy rate approaches zero may be overblown.

Our principal argument is as follows: The NK model is typically closed by a monetary policy rule in which the nominal short-term interest rate is the policy instrument of a monetary authority. The money market structure is treated as superfluous in this case. A simple rule assumes that interest rate deviations from steady state, i_t, respond to inflation deviations from steady state, π_t:

$$i_t = \phi\pi_t \tag{MP}$$

Model determinacy with rules-based interest rate policy requires that the Taylor principle, embedded in Taylor (1993), holds. The nominal rate must respond sufficiently to inflation fluctuations to generate countervailing real interest rate movements — in this case, $\phi > 1$.

We note, however, that in the background of the textbook NK model, the nominal interest rate must be consistent with equilibrium in the money market. In particular, it is possible to think about the *velocity of money* — the number of times a single money unit circulates within a period — as capturing the behavior of money demand. Rising (falling) money demand causes velocity to fall (rise). Higher nominal interest rates increase the opportunity cost of holding money, which reduces the demand for money and increases velocity. Lower nominal interest rates decrease the velocity of money. The velocity of money may also vary with other macroeconomic variables.

Suppose money velocity in the NK model is a function of the nominal interest rate and some additional variables, Z_t: $v_t = v(i_t; Z_t)$. It follows that the *quantity equation*, see Friedman (1970), can be written as:

$$m_t + v\left(i_t; Z_t\right) = p_t + y_t$$

where m_t is the money supply, p_t is the price level, $\pi_t = p_t - p_{t-1}$, and y_t, is aggregate output. The money supply and price level are expressed in log-deviations from trend. Money velocity and real output are expressed in log-deviations from steady state. A structural interpretation of velocity transforms the quantity equation from an accounting identity to an equilibrium condition. The nominal interest rate must be consistent with this equilibrium condition.

Imposing a specific interest rate policy rule, such as (MP), is isomorphic to imposing a money demand specification on the model. For example, if velocity growth responds to the nominal interest rate and output growth, $\Delta v_t = \varrho i_t + \Delta y_t$, the inverse money demand relationship implied by the quantity equation is observationally equivalent to (MP), absent money growth fluctuations:

$$i_t = \frac{1}{\varrho}\pi_t - \frac{1}{\varrho}\Delta m_t \tag{QE}$$

With exogenous money growth, the Taylor principle holds for $\varrho = \phi^{-1} < 1$, equivalent to the determinacy condition on (MP). If money growth responds to model variables itself, determinacy parameter restrictions and the model dynamics vary with the money supply policy.

Deriving an interest rate policy rule from the quantity equation is not novel. In fact, Taylor (1999) does so following similar logic to ours:

> *First imagine that the money supply is either fixed or growing at a constant rate. We know that velocity depends on the interest rate and on real output or income. Substituting for [velocity] in the quantity equation one thus gets a relationship between the interest rate, the price level, and real output. If we isolate the interest rate on the left-hand side of this relationship, we see a function of two variables: the interest rate as a function of the price level and real output. Shifts in this function would occur when either velocity growth or money growth shifts. Note also that such a function relating the interest rate to the price level and real output would still emerge if the money stock is not growing at a fixed rate, but rather responds in a systematic way to the interest rate or real output; the response of money will simply change the parameters in this relationship.*
>
> *– Taylor (1999, pp. 322-23)*

What is novel is taking the relationship between interest rate policy rules and the quantity equation seriously in the context of the NK model. Section 2 demonstrates that a general money demand specification is isomorphic to a general interest rate rule specification. Taylor rule-type estimates for the interest rate policy rule are indistinguishable from estimates from the money demand specification. Section 3 outlines the important implications of our alternative interpretation of the NK model. First, we derive a determinacy condition for money growth policy that nests the determinacy condition on conventional interest rate policy. Next, we derive the efficient path for money growth given a money demand specification. Finally, we prove that a determinate linear rational expectations equilibrium exists even under a permanent nominal interest rate peg and outline the limitations of policy in this case.

Under the money market interpretation of the model, the efficient money supply rule simultaneously stabilizes the output gap and inflation rate with the interest rate equal to the natural rate. These are the optimal dynamics in the textbook NK model and further point to the isomorphism between the model's rules-based interest rate policy and money market interpretations. The efficient money supply rule also solves an issue related to optimal policy implementation in the NK model — setting the nominal interest rate exogenously, e.g., equal to the natural rate, results in model indeterminacy. The efficient money supply rule generates money supply responses to shocks which imply that the nominal rate equals the natural rate. Although, when considering a fixed interest rate, a proxy for the PRLB constraint, the monetary authority cannot simultaneously target the output gap and inflation.

We perform all of the analysis in Section 3 in the context of the textbook NK model after outlining the textbook model for the reader's convenience below. A question that remains is if there is empirical support for our money market interpretation of the NK model. In Section 4, we estimate a generalized version of the textbook NK model with either conventional interest rate policy or our money market interpretation of the model, accounting for the binding PRLB constraint in the data. The non-linearity due to the potentially binding PRLB constraint requires us to account for this occasionally binding constraint when solving and filtering the model. We solve the model using the OccBin algorithm from Guerrieri and Iacoviello (2015). The OccBin algorithm constructs state-dependent piecewise-linear policy functions. In states in which the PRLB does not bind, the policy functions follow standard first-order perturbation solutions. In states in which the PRLB binds, the policy functions account for the fact that the policy rate is fixed, under the premise that the constraint will relax at some point in the future. The interaction of the state variables and expected binding constraint length can result in highly nonlinear dynamics. To account for this interaction and the non-linearity in the policy functions, we use the piecewise-linear Kalman filter from Giovannini et al. (2021) in our estimation exercises.

Estimating the model with conventional interest rate policy is similar in practice to the estimation exercises in Gust et al. (2017), Guerrieri and Iacoviello (2017), or Aruoba et al. (2021), for example. However, as noted in Hirose et al. (2022), estimating the textbook model through 2019 fails to generate a binding PRLB constraint period that matches the binding constraint duration in the data. With conventional interest rate policy, our estimates confirm this result. Hirose et al. (2022) introduce bounded rationality, as outlined in Gabaix (2020), into an otherwise standard rational expectations model following Atkinson et al. (2020) and show that this generates a persistent binding PRLB period consistent with the data. The non-monetary block in our estimated model also follows Atkinson et al. (2020), but with Calvo (1983) pricing rather than Rotemberg (1982) and steady state distortions due to non-zero steady-state net inflation. With our money market interpretation of the model, model estimates generate a long and persistent binding PRLB constraint period consistent with the data. That is, when interpreting the NK model through the lens of the money market, our estimates from an otherwise textbook NK model with rational expectations match the data, whereas the conventional interest rate policy estimates do not.

1.A The Textbook NK Model Structure

A first-order Taylor series approximation of the textbook NK model around the zero net inflation steady-state consolidates to two micro-founded relationships, the IS and Phillips curves:[1]

$$\text{(IS)} \qquad x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^*\right)$$

$$\text{(PC)} \qquad \pi_t = \kappa(\sigma + \eta) x_t + \beta \mathbb{E}_t \pi_{t+1}$$

where x_t is the output gap, i_t is the nominal short-term interest rate expressed in level deviations from steady-state, and π_t is the net inflation rate. The output gap is the log difference in output in an economy with monopolistic competition and nominal price rigidities in the production sector, and the same economy with flexible prices. The model parameters include the discount factor, β, inverse elasticity of intertemporal substitution, σ, inverse Frisch wage elasticity of labor supply, η, and output gap elasticity of inflation, $\kappa(\sigma + \eta)$.

The final term in the structural equations, r_t^*, is called the natural interest rate. The natural rate is the prevailing risk-free real short-term rate in the flexible price economy. To define the natural rate, consider the consumption Euler equation in the flexible price economy:

$$0 = \sigma\left(y_t^* - \mathbb{E}_t y_{t+1}^*\right) + r_t^*$$

Aggregate output in the flexible price economy, y_t^*, is proportional to a productivity shock, a_t:

$$y_t^* = \frac{1+\eta}{\sigma+\eta} a_t$$

Thus, assuming the productivity shock follows an AR(1) process with the AR coefficient ρ_a, the natural rate follows:

$$r_t^* = -\sigma(1-\rho_a)\frac{1+\eta}{\sigma+\eta} a_t$$

Aggregate output in the model follows:

$$y_t = x_t + \frac{1+\eta}{\sigma+\eta} a_t$$

For an overview of the literature on the NK model, see Goodfriend and King (1997) or Galí (2015, pp. 80–82). For quantitative exercises in Section 3, we assume that the structural parameters in the textbook model take on the following values: $\sigma = 1$, $\eta = 5$, $\beta = 0.99$, and $\kappa = 0.086$, consistent with the parameterization from Galí (2015, p. 67) assuming constant returns to scale over labor.

1. Appendix A derives the IS and Phillips curves from the underlying private economy equilibrium conditions including: the consumption Euler equation, $0 = \sigma\left(c_t - \mathbb{E}_t c_{t+1}\right) + i_t - \mathbb{E}_t \pi_{t+1}$; the labor supply curve, $w_t = \eta n_t + \sigma c_t$; the labor demand curve, $w_t = \mathrm{mc}_t + a_t$; the optimal price-setting condition, $\pi_t = \kappa \mathrm{mc}_t + \beta \mathbb{E}_t \pi_{t+1}$; the production function, $y_t = a_t + n_t$; and the aggregate resource constraint, $y_t = c_t$.

2 Money Demand with a Generalized Interest Rate Policy Rule

A general interest rate policy rule that nests many ad hoc interest rate policy specifications used in the NK literature can be written as:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left[\phi_\pi \pi_t + \phi_x x_t + \phi_y \Delta y_t + \phi_n (\pi_t + \Delta y_t)\right] + \epsilon_{i,t}$$

The interest rate responds to inflation, the output gap, real output growth, and nominal output growth with inertia, $\rho_i > 0$, where $\epsilon_{i,t}$ is a monetary policy shock.

Is there a money demand specification for which the endogenous interest rate response looks like the above interest rate policy rule absent monetary shocks? Yes! Consider the quantity equation in growth rates:

$$\Delta m_t + \Delta v_t = \pi_t + \Delta y_t$$

Let velocity growth follow:

$$\Delta v_t = \varrho (i_t + \omega \Delta i_t) + (\chi - \gamma)\Delta y_t - \xi x_t - \gamma \pi_t + \epsilon_{v,t}$$

where $\epsilon_{v,t}$ is a velocity growth shock, and substitute velocity from the quantity equation:[2]

$$i_t = \underbrace{\frac{\omega}{1+\omega}}_{\rho_i} i_{t-1} + \underbrace{\frac{1}{1+\omega}}_{1-\rho_i}\left[\underbrace{\frac{1}{\varrho}}_{\phi_\pi}\pi_t + \underbrace{\frac{\xi}{\varrho}}_{\phi_x} x_t + \underbrace{\frac{1-\chi}{\varrho}}_{\phi_y}\Delta y_t + \underbrace{\frac{\gamma}{\varrho}}_{\phi_n}(\pi_t + \Delta y_t)\right] \underbrace{- \frac{1}{1+\omega}\frac{1}{\varrho}(\Delta m_t + \epsilon_{v,t})}_{-\epsilon_{i,t}}$$

The interest rate rule and the inverse money demand equation have five parameters. Specifying an interest rate rule is equivalent to imposing a money demand specification. However, the interest rate rule is misspecified if we consider the money demand interpretation. The interest rate rule embeds the money growth term in the interest rate policy shock. Furthermore, it is clear from the derived money demand expression that estimating this equation, including money growth, implies that the residual from the estimation identifies the velocity growth shock.

Let us consider a specific velocity specification — velocity responds to observable series only, i.e., there is no output gap response, $\xi = 0$. Additionally, γ is not separately identifiable when estimating this equation. We estimate the following equation assuming $\gamma = 0$:

(2.1)
$$i_t = \frac{\omega}{1+\omega} i_{t-1} + \frac{1}{1+\omega}\left[\frac{1}{\varrho}\pi_t + \frac{1-\chi}{\varrho}\Delta y_t\right] - \frac{1}{1+\omega}\frac{1}{\varrho}\delta \Delta m_t + \tilde{\epsilon}_{i,t}$$

with $\delta = 0$ indicating conventional interest rate rule estimation and $\delta = 1$ implying the derived interest rate specification. We estimate equation (2.1) using data from 1984q1 to 2007q4. The observable series are the federal funds rate, inflation as measured by the GDP deflator, real GDP

2. Rearranging this equation defines a monetary base rule similar to McCallum (1988). However, as with an interest rate policy rule as outlined above, our exercise is akin to imposing a velocity structure on a McCallum-type rule.

Table 1: Interest Rate Policy versus Structural Money Demand Estimates

Interest rate rule: $\delta = 0$				*Structural money demand: $\delta = 1$*			
Structural		Reduced-form		Structural		Reduced-form	
ω	11.008 [3.540]	ρ_i	0.917 [0.025]	ω	18.700 [9.307]	ρ_i	0.949 [0.024]
ϱ	0.353 [0.111]	ϕ_π	2.830 [0.889]	ϱ	3.613 [3.132]	ϕ_π	0.277 [0.240]
χ	0.545 [0.154]	ϕ_y	1.285 [0.467]	χ	-6.283 [5.569]	ϕ_y	2.016 [1.070]

Notes: Estimation results from (2.1) using US data from 1984q1 to 2007q4. Bracketed numbers are the standard deviations of the point-estimates.

growth per capita, and monetary base growth per capita.

Table 1 provides the estimates. The structural parameter estimates are obtained via nonlinear regression techniques. We then compute the implied reduced-form parameters. Consider the interest rate rule results first, $\delta = 0$. In this case, the reduced-form estimates look like standard interest rate rule estimation results. However, with $\delta = 1$, the interest rate responses to inflation and output growth, ϕ_π and ϕ_y, are not well-identified.

Why are the estimates not identified when including money growth in the estimation specification? The estimation of the structural money demand function, $\delta = 1$, is an example of simultaneous equations estimation. In this case, we are estimating the equilibrium price in the money market, i_t, on a function of the equilibrium quantity, m_t. For example, suppose money growth co-moves with inflation and output growth:

$$\Delta m_t = \mu_\pi \pi_t + \mu_y \Delta y_t \tag{2.2}$$

and rearrange the estimation equation:

$$i_t = \frac{\omega}{1+\omega} i_{t-1} + \frac{1}{1+\omega}\left[\frac{1-\mu_\pi}{\varrho}\pi_t + \frac{1-\chi-\mu_y}{\varrho}\Delta y_t\right] + \widetilde{\epsilon}_{i,t} \tag{2.3}$$

Estimating (2.3) requires an identification restriction for (2.2) because μ_π and μ_y are not separately identifiable from ϱ and χ. Consider two identifying restrictions. The first is a crude version of the quantity theory — the growth rate of nominal income increases one-for-one with money growth ($\mu_\pi = \mu_y = 1$). The second is policy oriented — money growth varies inversely with nominal GDP growth. Under the quantity theory restriction, χ is not separately identifiable, so

Table 2: Structural Money Demand Estimates: Varying Identifying Restrictions

Quantity Theory: $\mu_\pi = \mu_y = 1$				*Policy:* $\mu_\pi = \mu_y = -0.5$			
Structural		Reduced-form		Structural		Reduced-form	
ω	19.778 [10.355]	ρ_i	0.952 [0.024]	ω	11.008 [3.540]	ρ_i	0.917 [0.025]
ϱ	0.502 [0.283]	ϕ_π	0	ϱ	0.530 [0.167]	ϕ_π	1.887 [0.593]
χ	-1	ϕ_y	1.992 [1.122]	χ	0.819 [0.232]	ϕ_y	0.342 [0.408]

Notes: Estimation results from (2.3) with varying identification restrictions using US data from 1984q1 to 2007q4. Bracketed numbers are the standard deviations of the point-estimates.

we estimate ω and ϱ conditional on a value of χ.

Table 2 provides the estimates of equation (2.3) with either identifying restriction. Under the quantity theory, the interest rate response to inflation is assumed to be 0. The output gap response is, again, not well-identified. Now, consider the policy-oriented identifying restriction — the second set of results in Table 2 with $\mu_\pi = \mu_y = -0.5$. In this case, the reduced-form estimates are in line with those typically used in the NK literature, $\phi_\pi > 1$ and $\phi_y \approx 0$.

In summary, rules-based interest rate policy specifications are not separately identifiable from money demand specifications. Estimating a typical interest rate policy rule is equivalent to estimating an inverse money demand function, except that the interest rate rule pushes money growth variability into the residual. The interest rate is the equilibrium price in the money market and, thus, a function of the quantity of money, implying interest rate policy rules are not identified. Identification requires assuming a money supply specification.

With a money demand specification that is observationally equivalent to a rules-based interest rate policy that targets inflation and output growth, money growth must fall in response to either inflation or output growth to yield estimates consistent with specifications used in the NK literature. It is important to note that this is consistent with casual empirical observation. For example, prior to the payment of interest on reserves, the target for the federal funds rate was maintained through open market operations. Changes in the stance of monetary policy caused by higher inflation would bring about open market sales, reducing money growth and raising the federal funds rate target. Even in the period following interest payment on reserves, tightening monetary policy in response to higher inflation entails raising the interest rate and quantitative tightening. Thus, casual empirical observation does not help us distinguish between the two competing interpretations of the model.

3 The Effect of the Money Market on the NK Model

In an otherwise neoclassical model, the NK model introduces two modeling features to generate a relatively elastic aggregate supply curve (the AS curve in output-inflation space is upward sloping, but not vertical). The first is monopolistic competition which introduces an optimal pricing problem in the production sector. The optimal pricing problem allows for the second modeling feature, nominal price rigidity in goods prices. Nominal price rigidity causes the aggregate supply curve to be relatively elastic compared to the neoclassical model, in which aggregate supply is perfectly inelastic. These two modeling features generate a time-varying wedge between factor input demand from the production sector and factor input supply, i.e., a markup in the labor market. This time-varying markup causes the *classical dichotomy* to fail. Output dynamics vary with the stance of monetary policy. The time-varying markup generates inefficient labor demand shifts. Monetary policy offsets these effects by shifting household labor supply.

The *classical dichotomy* is the idea that nominal variables have no effect on real outcomes. In general, the classical dichotomy does not hold in the NK model. To see this, consider the Phillips curve, equation (PC), written in terms of output:

$$y_t = \frac{1+\eta}{\sigma+\eta} a_t - \frac{1}{\kappa(\sigma+\eta)} \left(\beta \mathbb{E}_t \pi_{t+1} - \pi_t\right) \tag{3.1}$$

With flexible prices, $\kappa \to \infty$, it is clear that the classical dichotomy holds. Aggregate output fluctuations are proportional to productivity and, from the IS curve, the ex ante real short-term rate equals the natural rate. With sticky prices, i.e., finite κ, this is not the case.

For example, consider the velocity growth specification that implies the interest rate dynamics follow equation (QE), and substitute current and future inflation from equation (3.1):

$$y_t = \frac{1+\eta}{\sigma+\eta} a_t - \frac{1}{\kappa(\sigma+\eta)} \left(\beta \mathbb{E}_t \Delta m_{t+1} - \Delta m_t + \varrho \left(\beta \mathbb{E}_t i_{t+1} - i_t\right)\right)$$

Output varies with productivity and the stance of monetary policy. A solution to the model exists conditional on a money supply policy. The classical dichotomy fails absent: $\Delta m_t = -\varrho i_t$.

Proposition 1. *In a NK model with money demand that is observationally equivalent to inflation-targeting interest rate policy, absent money growth variability, and money growth policy that responds to inflation and the output gap: $\Delta m_t = \mu_\pi \pi_t + \mu_x x_t$; a determinate linear rational expectations equilibrium exists if:*

$$\mu_x < \frac{\kappa(\sigma+\eta)}{1-\beta}(1-\varrho-\mu_\pi)$$

Proof. See Appendix B. □

This condition nests the Taylor principle, $\varrho < 1$, with fixed money growth, $\mu_x = \mu_\pi = 0$. With no response of money growth to the output gap, $\mu_x = 0$, this condition simplifies to $\varrho < 1 - \mu_\pi$.

3.A Monetary Policy with Money

The textbook NK model imposes a rules-based interest rate policy on the IS and Phillips curves. We have shown that rules-based interest rate policies are observationally equivalent to imposing money velocity specifications in the quantity equation, implying a money demand structure. An important question is whether or not this has implications for the operation and interpretation of monetary policy. We examine this in two respects. First, we consider the case where our alternative money market interpretation of the NK model is correct and derive the efficient money supply policy in the model. Second, we consider the case where the nominal interest rate is permanently fixed, a proxy for a PRLB constraint. We show that even with the interest rate permanently fixed, a determinate linear rational expectations equilibrium exists. However, it is impossible to simultaneously target inflation and the output gap in this case.

1. An Efficient Money Supply Rule Suppose the goal of monetary policy is to maximize household welfare. The output gap is fixed under the efficient, or welfare maximizing policy. From the Phillips curve, inflation is permanently fixed when the output gap is fixed. From the IS curve, the interest rate equals the natural rate in this case. As shown in Section 3, the flexible price, or efficient, output response is proportional to productivity:

$$y_t = \frac{1+\eta}{\sigma+\eta} a_t$$

What money supply rule supports the welfare maximizing dynamics in equilibrium?

Consider the general velocity specification from Section 2 and the quantity equation in growth rates under the efficient dynamics (output is proportional to productivity, $r_t = r_t^*$, and $\pi_t = 0$):

$$\Delta v_t = \varrho\left(r_t^* + \omega \Delta r_t^*\right) - \frac{\chi - \gamma}{\sigma(1-\rho_a)} \Delta r_t^* + \epsilon_{v,t}$$
$$\Delta v_t = -\frac{1}{\sigma(1-\rho_a)} \Delta r_t^* - \Delta m_t$$

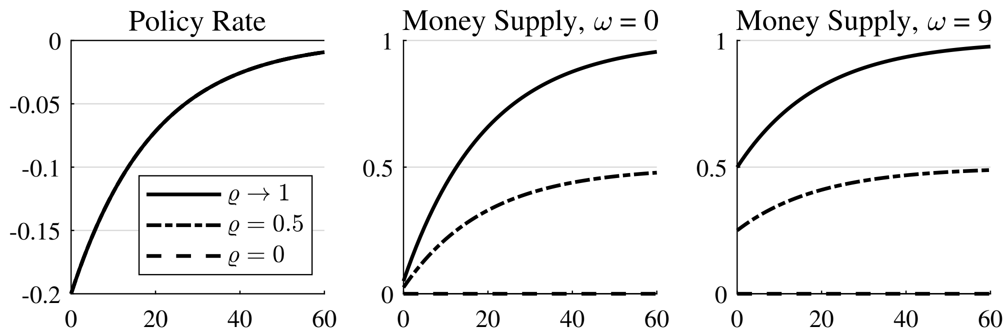
Combining these two equations defines the efficient money growth policy:

$$\Delta m_t = \varrho\sigma(1-\rho_a)\frac{1+\eta}{\sigma+\eta} a_t + \left[1+\gamma-\chi+\varrho\omega\sigma(1-\rho_a)\right]\frac{1+\eta}{\sigma+\eta}\Delta a_t - \epsilon_{v,t} \tag{3.2}$$

With a velocity specification that yields observationally equivalent model dynamics to rules-based interest rate policy, the optimal money growth rule implies money is inelastically supplied (does not vary with the policy rate) and responds to productivity fluctuations. Money growth responds to the productivity level, with productivity growth (contractions) amplifying (dampening) the money growth response.[3] The size of the shift in the money supply curve following

3. Due to nominal price ridigities, expansionary productivity shocks generate a negative output gap. Output expands less following productivity shocks than it otherwise would in a model with flexible prices, generating a

Figure 1: Efficient Monetary Policy Responses to a Productivity Shock, a_t



Notes: Impulse responses to a productivity shock under the efficient money growth rule, equation (3.2), varying the interest rate semi-elasticity of money demand, ϱ, with $\gamma = 0$ and $\chi = 1$. Solid lines: $\varrho \approx 1$; dashed-dotted lines: $\varrho = 0.5$; dashed lines: $\varrho = 0$. The interest rate response is in level deviations from steady state in annualized percentages and the interest rate equals the natural rate across all interest rate semi-elasticity of money demand values. The money supply response is in percentage deviations from steady state.

a productivity shock varies with the structural money demand parameters, the persistence of the shock process, and household preferences. Again, this is the optimal money growth policy with a velocity specification such that the quantity equation is observationally equivalent to the majority of interest rate policy rules used in the NK literature. What matters for optimal policy is correctly identifying the output growth elasticity of money demand, $1 + \gamma - \chi$, and the parameters defining the interest rate semi-elasticity of money demand, $\varrho(1 + \omega)$.

Figure 1 shows the efficient money supply response to the persistent productivity shock ($\rho_a = 0.95$).[4] The left panel shows the policy rate response. The policy rate equals the natural rate *regardless of the money demand specification*. The center panel shows the money supply response with $\omega = \gamma = 0$ and $\chi = 1$, varying the interest rate semi-elasticity of money demand, ϱ. When money demand varies with the interest rate, $\varrho > 0$, the money supply grows following a productivity shock. If money demand does not vary with the interest rate, $\varrho = 0$, holding the money supply fixed provides the efficient outcome. The right panel shows the same responses with $\omega = 9$ implying an AR coefficient on the interest rate in the economy of 0.9. The inertia in the interest rate dynamics amplifies the impact response of money growth to the productivity shock. In either case, the money growth response is quite persistent given the persistence of the productivity shock. 5 years after the shock (20 quarters) the money supply is still growing substantially and does not converge to its new steady-state level until after 12 years (48 quarters).

Again, the policy rate equals the natural rate across all money market specifications. That is, the optimal money supply rule ensures that the technically efficient policy rate dynamics

negative output gap.

4. Note that the response of the money supply to a productivity shock is positive because a productivity shock causes potential GDP to increase by more than actual real GDP in the NK model. This implies that the response of the output gap to a productivity shock is negative.

hold. Simply setting the policy rate equal to the natural rate in the NK model is not possible as exogenously setting the policy rate results in model indeterminacy. This relationship is an equilibrium outcome when money supply policy follows equation (3.2).

2. Money Supply Rules with a Fixed Interest Rate What money supply specifications support a permanently fixed interest rate? Consider, again, the general velocity specification from Section 2:

$$\Delta v_t = \varrho\left(i_t + \omega \Delta i_t\right) + (\chi - \gamma)\Delta y_t - \xi x_t - \gamma \pi_t + \epsilon_{v,t}$$

We fix the interest rate at $i_t \equiv 0$ for all t. Substituting velocity growth into the quantity equation yields:

$$\Delta m_t = (1+\gamma)\pi_t + (1+\gamma-\chi)\Delta y_t + \xi x_t - \epsilon_{v,t} \tag{3.3}$$

Equations (3.3), (IS), and (PC) define an equilibrium with a fixed interest rate, $i_t \equiv 0$, given a money growth policy. The IS and Phillips curves combine into a single equation:

$$\left(\beta\sigma + \kappa(\sigma+\eta)\right) x_t = \beta\sigma \mathbb{E}_t x_{t+1} + \pi_t + \beta r_t^* \tag{3.4}$$

Proposition 2. *In a NK model with money demand that is observationally equivalent to rules-based interest rate policy that responds to inflation and the output gap, $\gamma = 0$ and $\chi = 1$, absent money growth variability, and a money growth policy that responds to inflation and the output gap: $\Delta m_t = \mu_\pi \pi_t + \mu_x x_t$; a determinate linear rational expectations equilibrium with a fixed interest rate, $i_t \equiv 0$, exists if:*

$$\mu_x < \kappa(\sigma+\eta)(1-\mu_\pi) + \xi$$

Proof. Substitute the money supply policy specification from equation (3.3) with $\gamma = 0$ and $\chi = 1$:

$$\mu_\pi \pi_t + \mu_x x_t = \pi_t + \xi x_t - \psi_t \Rightarrow \pi_t = \frac{\mu_x - \xi}{1-\mu_\pi} x_t + \frac{1}{1-\mu_\pi}\epsilon_{v,t}$$

Substituting inflation from equation (3.4):

$$\left[\left(\beta\sigma + \kappa(\sigma+\eta)\right)(1-\mu_\pi) + \xi - \mu_x\right] x_t = \beta\sigma(1-\mu_\pi)\mathbb{E}_t x_{t+1} + \epsilon_{v,t} + \beta(1-\mu_\pi) r_t^*$$

which has a determinate solution given the above condition on μ_x is satisfied. □

The condition on μ_x will always hold for standard parameterizations of a money growth rule. One would expect that the central bank would respond to rising inflation and a positive output gap with lower money growth. This implies that $\mu_x < 0$ and $\mu_\pi < 0$. If $\mu_\pi \leq 0$, then the term of the right-hand side of the condition is positive. Therefore, the condition holds for any $\mu_x \leq 0$.

Corollary 2.1. *Output gap-targeting money supply policy with a fixed interest rate, $i_t \equiv 0$, responds* *to current and past productivity shocks, and current velocity growth shocks, expanding with a positive productivity shock.*

Proof. Equation (3.3) allows inflation to be substituted from the problem, defining an equilibrium over output and the money supply, given a money supply policy specification:

$$(\beta\sigma + \kappa(\sigma+\eta))\, x_t = \beta\sigma \mathbb{E}_t x_{t+1} + \frac{1}{1+\gamma}\left(\Delta m_t + \epsilon_{v,t}\right) - \frac{1+\gamma-\chi}{1+\gamma}\Delta y_t - \frac{\xi}{1+\gamma} x_t + \beta r_t^*$$

Targeting the output gap implies output is proportional to productivity. Rewrite the equilibrium condition with a fixed output gap to define output gap targeting money supply policy:

$$\Delta m_t = \beta(1+\gamma)\sigma(1-\rho_a)\frac{1+\eta}{\sigma+\eta} a_t + (1+\gamma-\chi)\frac{1+\eta}{\sigma+\eta}\Delta a_t - \epsilon_{v,t}$$

□

Corollary 2.2. *Inflation-targeting money supply policy with a fixed interest rate, $i_t \equiv 0$, responds to current and past productivity shocks, and current velocity growth shocks, contracting with a positive productivity shock so long as:* $(1-\gamma-\chi)\kappa(\sigma+\eta) < \xi\beta\sigma(1-\rho_a)$.

Proof. Consider equation (3.4) with inflation targeting money supply policy:

$$(\beta\sigma + \kappa(\sigma+\eta))\, x_t = \beta\sigma \mathbb{E}_t x_{t+1} + \beta r_t^* \Rightarrow y_t = \frac{\kappa(1+\eta)}{\beta\sigma(1-\rho_a) + \kappa(\sigma+\eta)} a_t$$

Substitute output from equation (3.3):

$$\Delta m_t = -\xi \frac{\beta\sigma(1-\rho_a)}{\beta\sigma(1-\rho_a)+\kappa(\sigma+\eta)}\frac{1+\eta}{\sigma+\eta} a_t + (1+\gamma-\chi)\frac{\kappa(1+\eta)}{\beta\sigma(1-\rho_a)+\kappa(\sigma+\eta)}\Delta a_t - \epsilon_{v,t}$$

□

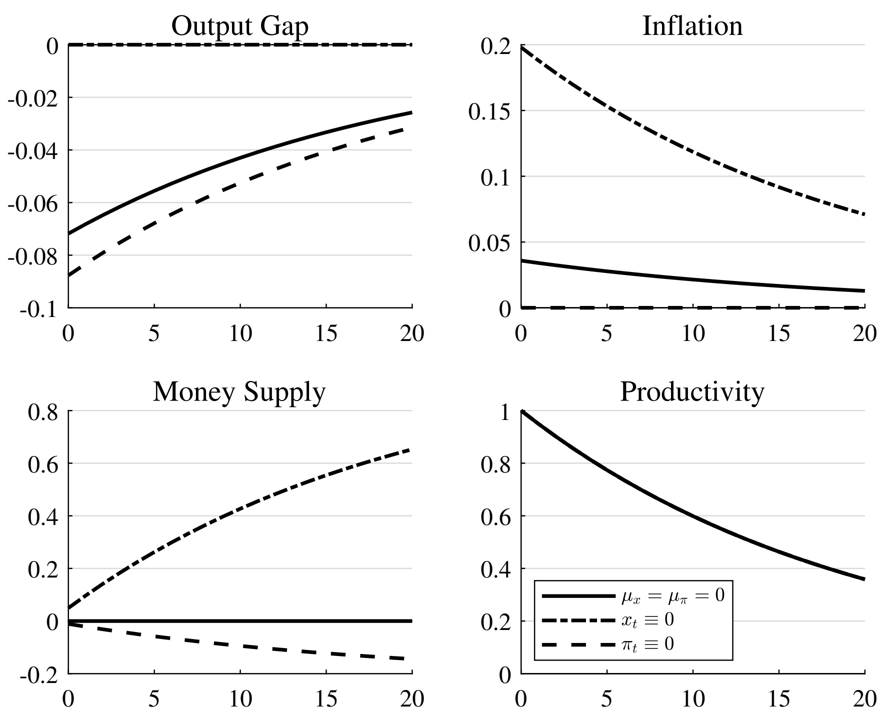
Figure 2 shows the impulse responses to a productivity shock with fixed, output gap-targeting, and inflation-targeting money supply policy. Fixed and output gap-targeting money supply policy cause the productivity shock to be inflationary with a fixed nominal interest rate. Fixed and inflation-targeting money supply policy generate a negative output gap response with a fixed nominal interest rate.

Proposition 3. *It is impossible to simultaneously target inflation and the output gap with money supply policy supporting a fixed interest rate, $i_t \equiv 0$, — the efficient, or welfare maximizing, outcome.*

Proof. Consider output gap-targeting money supply policy, implying from (3.4):

$$\pi_t = \beta\sigma(1-\rho_a)\frac{1+\eta}{\sigma+\eta} a_t$$

Figure 2: Impulse Responses to a Productivity Shock, a_t, with $i_t \equiv 0$



Notes: Impulse responses to a 1% expansionary productivity shock with a fixed interest rate, $i_t \equiv 0$. Solid lines: output-gap-targeting money supply policy; dashed-dotted lines: inflation-targeting money supply policy; dashed lines: rules-based money supply policy with $\mu_x = \mu_\pi = 0$ (fixed money supply). The output gap, money supply, and productivity responses are in percentage deviations from steady state. The inflation responses are level deviations from steady state in annualized percentages. $\chi = 1$, $\gamma = 0$, and $\zeta \approx 0.125$

Inflation is proportional to the productivity shock when considering output gap-targeting money supply policy with a fixed interest rate. A similar argument follows when considering inflation-targeting money supply policy in equation (3.4). □

Overall, the results from this section have important implications for both the conduct and interpretation of monetary policy. First, it is possible to specify the efficient path of the money supply that is consistent with a corresponding efficient path of the nominal interest rate. Thus, our money market interpretation of the NK model allows one to think about monetary policy in terms of the money supply. This is particularly important given the issues related to the PRLB constraint. This lower bound has given rise to practical concerns about the effectiveness of monetary policy when the policy rate approaches zero. In addition, concerns with this lower bound have also raised concerns about the determinacy of equilibria. Our results demonstrate that concerns about indeterminacy might be overblown, considering that there is a determinate rational expectations equilibrium even with a permanently fixed nominal interest rate. Nonetheless, our results do speak to a limitation of policy. With a fixed nominal interest rate, the central bank cannot target *both* inflation and the output gap.

4 Empirical Evidence: General Equilibrium Estimation 1984 – 2019

In this section we consider the effects of our money market interpretation of the NK model on model estimates of the general equilibrium model. We estimate the model using standard Bayesian techniques. To account for the binding PRLB constraint in the data in our estimation, we solve the model using the OccBin solution algorithm from Guerrieri and Iacoviello (2015) and use the piecewise-linear Kalman filter from Giovannini et al. (2021).

The model we estimate includes three extensions relative to a generalized version of textbook NK model outlined in Section 1 that follow the NK model estimation literature (see Aruoba et al. (2021) or Atkinson et al. (2020), for example). First, we add trend productivity growth with shocks to productivity growth, rather than stationary productivity shocks. This accounts for: (i) trend output growth in the data; and (ii) the persistent deviation in output from trend beginning in the third quarter of 2008. Second, we add habit formation in consumption. Finally, we add preference shocks that change the weight households put on utility flows in a given period. The preference shock acts as a demand shock on the economy. When households weight current utility flows higher than future utility flows their stochastic discount factor is exogenously reduced, shifting expenditures to the current period. We generalize the model relative to Section 1 by considering a (potentially) non-zero net inflation steady state with Calvo (1983) pricing.

The non-monetary model block is identical across estimation specifications and includes eight structural equations:[5]

$$(4.1)\qquad (z-h)\,\lambda_t = -\sigma\left(zy_t - hy_{t-1} + hz_t\right)$$

$$(4.2)\qquad w_t = \eta\left(\delta_t + y_t\right) - \lambda_t$$

$$(4.3)\qquad i_t = \left(\psi_t - \mathbb{E}_t\psi_{t+1}\right) - \left(\mathbb{E}_t\lambda_{t+1} - \lambda_t\right) + \mathbb{E}_t z_{t+1} + \mathbb{E}_t\pi_{t+1}$$

$$(4.4)\qquad \pi_{\#,t} = \pi_t + g_t - h_t$$

$$(4.5)\qquad g_t = \left(1-\beta\phi\Pi^{\varepsilon}\right)\left(\psi_t + w_t + y_t + \lambda_t\right) + \beta\phi\Pi^{\varepsilon}\left(\varepsilon\mathbb{E}_t\pi_{t+1} + \mathbb{E}_t g_{t+1}\right)$$

$$(4.6)\qquad h_t = \left(1-\beta\phi\Pi^{\varepsilon-1}\right)\left(\psi_t + y_t + \lambda_t\right) + \beta\phi\Pi^{\varepsilon-1}\left((\varepsilon-1)\mathbb{E}_t\pi_{t+1} + \mathbb{E}_t h_{t+1}\right)$$

$$(4.7)\qquad \pi_t = (1-\phi)\left(\frac{\Pi}{\Pi_{\#}}\right)^{\varepsilon-1}\pi_{\#,t}$$

$$(4.8)\qquad \delta_t = \varepsilon\pi_t - \varepsilon\left(1-\phi\Pi^{\varepsilon}\right)\pi_{\#,t} + \phi\Pi^{\varepsilon}\delta_{t-1}$$

along with two exogenous processes:

$$(4.9)\qquad z_t = \rho_z z_{t-1} + \sigma_z\epsilon_{z,t}$$

$$(4.10)\qquad \psi_t = \rho_\psi\psi_{t-1} + \sigma_\psi\epsilon_{\psi,t}$$

where all variables are expressed in log-deviations from steady state.

Equation (4.1) defines the marginal utility of current consumption, λ_t, given the addition of habit formation, $h > 0$, trend productivity growth, $z > 1$, and the productivity growth shock, z_t. Equation (4.2) is household labor supply where η is the inverse wage elasticity, or Frisch elasticity, of labor supply. Equation (4.3) is the Fisher equation. The real interest rate includes the inverse intertemporal rate of substitution over household consumption, the effect of varying preferences over utility flows across time, $\psi_t - \mathbb{E}_t\psi_{t+1}$, and expected future productivity growth. Equations (4.4) – (4.6) define the optimal reset price for the fraction $1-\phi$ of firms that reset prices in the current period, relative to the aggregate price level in the prior period where aggregate inflation follows equation (4.7). β is the household discount factor and ε is the goods elasticity of substitution which governs the average markup in the economy. Equation (4.8) defines a measure of price dispersion in the economy reflecting that $(1-\phi)$ firms optimally reset prices in the current period and ϕ firms hold prices constant. Equations (4.9) and (4.10) imply that productivity growth and the intertemporal preference shifter follow AR(1) processes in logs. Note, the steady state optimal reset price, relative to the price level in the prior period, varies with the steady state inflation rate in the economy: $\Pi_{\#} = \left(\left(\Pi^{1-\varepsilon}-\phi\right)/(1-\phi)\right)^{1/(1-\varepsilon)}$. Equations (4.1) – (4.8) consolidate to the NK model outlined in Section 1 with productivity growth (rather than stationary productivity) and preference shocks with: $z = \Pi = 1$ and $h = 0$, where $\kappa = (1-\phi)(1-\phi\beta)/\phi$.

5. Appendix C provides micro-foundations for equations (4.1) – (4.8). We derive each from optimization problems for households and firms, impose an equilibrium, and log-linearize the nonlinear and stationary equilibrium conditions around a non-zero net inflation steady state.

We estimate the exogenous process parameters, $\{\rho_z,\ \rho_\psi,\ \sigma_z,\ \sigma_\psi\}$, in addition to the degree habit formation, h, degree of price rigidity, ϕ, average productivity growth, z, and average gross inflation, Π, given calibrated values for the remaining parameters and a specification of the monetary block in the economy. The intertemporal elasticity of substitution, $1/\sigma$ is 0.5. The household discount factor, β, equals 0.995. The Frisch elasticity, $1/\eta$, equals $1/3$, implying that the labor supply curve is relatively wage inelastic. The goods elasticity of substitution, ε, is 7.667, implying a 15% average markup in the zero net inflation steady state.

We consider two specifications of the monetary block in the economy. The first is conventional interest rate policy. In this case, the nominal interest rate is subject to a PRLB constraint:

$$i_t = \max\left\{i_t^n, -\left(1-\beta^{-1}\Pi z\right)\right\} \tag{4.11}$$

where i_t^n is the notional rate and the interest rate level cannot fall below zero ($\beta^{-1}\Pi Z$ is the steady state gross nominal interest rate). The notional rate, i_t^n, responds to inflation, output growth, and nominal output growth deviations from trend with inertia, and we introduce a monetary policy shock, $\epsilon_{i,t}$:

$$i_t^n = \rho_i i_{t-1}^n + (1-\rho_i)\left[\phi_\pi \pi_t + \phi_y\left(y_t - y_{t-1} + z_t\right) + \phi_n\left(\pi_t + y_t - y_{t-1} + z_t\right)\right] + \sigma_i \epsilon_{i,t} \tag{4.12}$$

An equilibrium in this case is defined by equations (4.1) – (4.12) and includes three shocks. Thus, we include three observable series, all of which are included in the alternative monetary block specification estimation exercise discussed below. The observable series are: (i) real GDP growth per capita, where population growth is calculated using an eight quarter moving average as in Aruoba et al. (2021); (ii) GDP deflator implied quarterly inflation; and (iii) the quarterly effective federal funds rate. The PRLB binds beginning in the fourth quarter of 2008 and "lifts-off" in the fourth quarter of 2015. During this period, the monetary policy shock drops from the model. Given this, we censor the federal funds rate data during this period as missing to avoid stochastic singularity due to the number of observable series exceeding the number of shocks. That is, the model dictates the paths of the notional and policy rates from 2008q4 through 2015q3.

The second monetary specification allows for our money market reinterpretation of interest rate policy. In this case, the policy rate is subject to the same PRLB constraint, but the "policy parameters" in the notional rate rule are interpreted as structural money market parameters:

$$i_t^n = \underbrace{\frac{\omega}{1+\omega}}_{\rho_i} i_{t-1}^n + \underbrace{\frac{1}{1+\omega}}_{1-\rho_i}\left[\underbrace{\frac{1+\gamma}{\varrho}}_{\phi_\pi+\phi_n}\pi_t + \underbrace{\frac{1-\chi+\gamma}{\varrho}}_{\phi_y+\phi_n}\left(y_t - y_{t-1} + z_t\right)\right] - \overbrace{\underbrace{\frac{1}{(1+\omega)\varrho}}_{(1-\rho_i)\phi_\pi}\left(\Delta m_t + \epsilon_{\psi,t}\right)}^{-\sigma_i\epsilon_{i,t}} \tag{4.13}$$

where $\epsilon_{\psi,t}$ is a money velocity, or demand, shock and money growth follows:

$$\Delta m_t = \rho_m \Delta m_{t-1} + \sigma_m \epsilon_{m,t} \tag{4.14}$$

An equilibrium is now defined by equation (4.1) – (4.11), (4.13), and (4.14). Introducing the money growth shock requires a fourth observable. We use the quarterly per capita growth rate of the monetary base as the observable for money growth in the model.[6] Again, at the PRLB, we censor the federal funds rate data as the money velocity shock now drops from the model. The model dictates the path of the nominal and notional interest rates as in the conventional case. However, in this case, equation (4.14) captures the effects of money growth on the notional rate. Monetary base growth at the PRLB in the data causes the notional rate to fall.

All data is in net percentage levels. The measurement equation for estimation follows:

$$\begin{bmatrix} 1+\text{GDPGrowth}_t/100 \\ 1+\text{DefInflation}_t/100 \\ 1+\text{FedFunds}_t/100 \\ 1+\text{MBGrowth}_t/100 \end{bmatrix} = \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & \beta^{-1}\Pi Z & 0 \\ 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} (y_t - y_{t-1} + z_t) \\ \pi_t \\ r_t \\ \Delta m_t \end{bmatrix} + \begin{bmatrix} Z \\ \Pi \\ \beta^{-1}\Pi Z \\ G \end{bmatrix}$$

where the monetary base growth observable and all money growth related parameters are omitted in the conventional estimation. We set the federal funds rate to missing from 2008q4 through 2015q3 in both estimation exercises. G is the steady state gross money growth rate. Table 3 provides the calibrated parameter values, prior distributions for the estimated parameters, and estimated posterior distributions under each monetary block specification.

We estimate the model under each specification in two steps. First, we compute estimates of the posterior mode and parameter covariance matrix using Monte Carlo optimization. Next, we generate a Monte Carlo Markov Chain (MCMC) of 500,000 parameter draws initialized at this mode using the Metropolis-Hastings sampler. We report the posterior mean, median, and 90% confidence interval for parameters from the MCMC, dropping the first 125,000 draws. We tune the j-scale parameter prior to generating the MCMC to target an acceptance ratio of 33%. The acceptance ratio for the MCMC from the conventional estimation is 33.2% and 31.5% in the money market estimation. Table 3 presents the prior distributions and estimated posterior distributions of all estimated parameters under each specification.

The conventional estimates are consistent with the past literature. Prices are fairly sticky with an implied average fixed price duration across firms of approximately 7 quarters at the posterior mean. There is modest habit formation ($h \approx 0.21$). Interest rate policy responds more sensitively to inflation ($1.3 + 1.6 = 2.9$) than real output growth ($1.3 + 0.6 = 1.9$). The mean annual productivity growth rate is 1.7%. The mean annual inflation rate is 3.4%.

The money market estimates differ from the conventional estimates. Prices are nearly four times as sticky with an implied average fixed price duration across firms of approximately 26 quarters at the posterior mean. There is over three times as much habit formation. Interest

6. Monetary base growth exhibits significant seasonality prior to the beginning of the Fed QE programs in 2008. For this reason, we seasonally adjust the monthly monetary base series up to September 2008. We then keep the end-of-quarter monthly level of the seasonally adjusted data and merge this with the raw end-of-quarter monetary base measures beginning in 2008q4. We compute the growth rate of this monetary base variable. We then convert this to per capita terms by subtracting off the 8-quarter moving average of population growth as with real GDP growth.

Table 3: Calibrated and Estimated Parameter Values

	Calibrated Parameters	Value
β	Household discount factor	0.995
η	Inverse Frisch elasticity	3
σ	Inverse elasticity of intertemporal substitution	2
ε	Goods elasticity of substitution	7.667

	Parameter	Prior [mean, std]	Posterior							
			Conventional Estimation				Money Market Estimation			
			Mean	5%	Median	95%	Mean	5%	Median	95%
ϕ	Calvo parameter	beta [0.5, 0.2]	0.8572	0.8171	0.8597	0.8988	0.9620	0.9524	0.9620	0.9716
h	Habit in consumption	beta [0.5, 0.2]	0.2138	0.0809	0.2072	0.3435	0.6848	0.5951	0.6884	0.7708
ρ_i	Notional rate inertia	beta [0.5, 0.2]	0.8517	0.8161	0.8528	0.8880	0.9247	0.9013	0.9255	0.9500
ϕ_π	Inflation response	normal [1, 0.5]	1.6142	1.0535	1.5995	2.1760	0.6684	0.5354	0.6671	0.7952
ϕ_y	Real growth response	normal [1, 0.5]	0.6374	-0.0100	0.0666	1.2538	0.7289	0.3103	0.7574	1.0363
ϕ_n	Nominal growth response	normal [1, 0.5]	1.2561	0.6683	1.2569	1.8175	1.6968	1.1389	1.6549	2.5097
ρ_z	AR(1) productivity growth	beta [0.5, 0.2]	0.4029	0.1364	0.4000	0.6964	0.0706	0.0046	0.0576	0.1397
ρ_ν	AR(1) preference shock	beta [0.5, 0.2]	0.9524	0.9318	0.8542	0.9736	0.9668	0.9529	0.9681	0.9825
σ_z	Std. productivity growth	invgamma [0.01, 0.01]	0.0069	0.0045	0.0067	0.0091	0.0235	0.0165	0.0249	0.0274
σ_ν	Std. preference shock	invgamma [0.01, 0.01]	0.0414	0.0268	0.0399	0.0555	0.0772	0.0582	0.0778	0.0941
σ_i	Std. monetary policy shock	invgamma [0.01, 0.01]	0.0018	0.0016	0.0018	0.0020	0.0143	0.0034	0.0103	0.0289
$100(Z-1)$	Net productivity growth	normal [1, 0.33]	0.4258	0.2750	0.4249	0.5778	0.4913	0.2423	0.4756	0.8038
$100(\Pi-1)$	Net inflation rate	normal [1, 0.33]	0.8372	0.6961	0.8352	0.9880	0.1110	-0.0837	0.0344	0.4024
$100(G-1)$	Net money growth	normal [1.5, 0.5]	—	—	—	—	1.1044	0.6861	1.0871	1.4650
ρ_m	AR(1) money growth	beta [0.5, 0.2]	—	—	—	—	0.1233	0.0426	0.1207	0.1980
σ_m	Std. money growth shock	invgamma [0.01, 0.01]	—	—	—	—	0.0750	0.0680	0.0745	0.0850
σ_v	Std. money velocity shock	invgamma [0.01, 0.01]	—	—	—	—	0.0388	0.0289	0.0378	0.0488

rate policy approximately follows a nominal GDP target ($\phi_\pi + \phi_n \approx \phi_y + \phi_n \approx 1.7 + 0.7 = 2.4$) with a more persistent interest rate rule than the conventional estimates. The mean annual productivity growth is 2.0% and mean annual inflation is 0.4%. The implied monetary policy shock variability, σ_i, is nearly eight times higher under the money market specification of the model than the conventional estimates.

Figure 3 plots the data and filtered levels of the observable series at the posterior median across the two monetary block specifications. As documented in Hirose et al. (2022), the conventional estimation exercise fails to match the duration of the binding PRLB constraint. The notional rate returns to near-zero in 2011, generating a preemptive and persistent lift-off in the model by the end of 2011, contrary to the data. Under the money market interpretation of the model with observable money growth, the model generates a long and persistent binding PRLB constraint that matches the data. Furthermore, our estimates are consistent with counterfactual federal funds rate estimates from Rich (2022).[7]

5 Conclusion

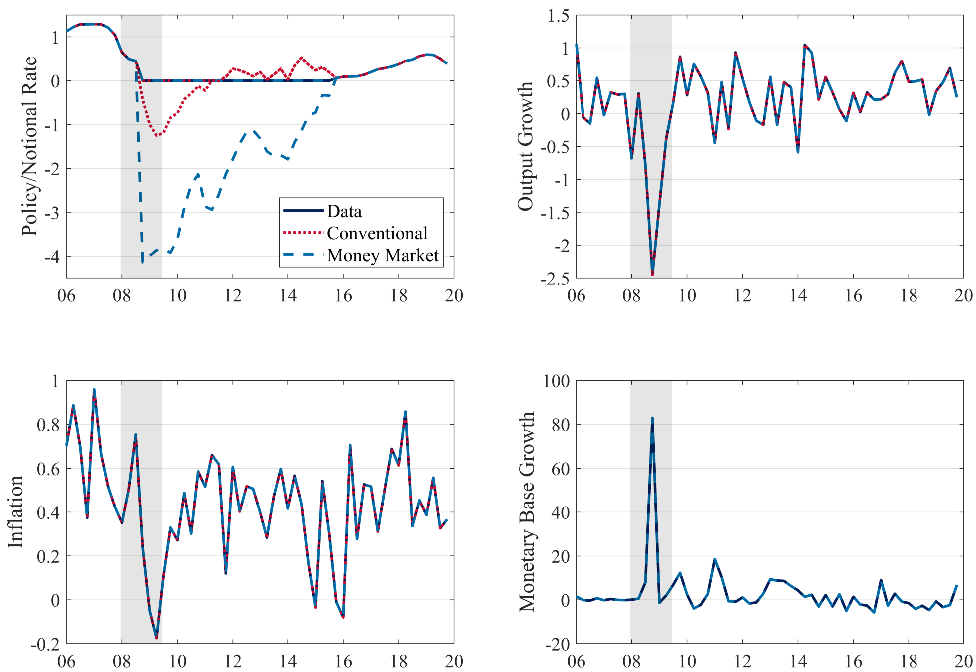
The NK model is central to current monetary policy and business cycle analysis. Researchers close this model by imposing an interest rate rule for monetary policy. This paper has provided an alternative interpretation of the NK model that treats the interest rate equation as an equilibrium condition derived from a money demand specification. The implications are significant. In a world characterized by so-called unconventional policies consisting mainly of large-scale asset purchases and the nominal interest rate stuck near the zero lower bound, it might be helpful to think about monetary policy in terms of the path of money growth. In that context, there is nothing unconventional about large-scale asset purchases except potentially for the magnitude.

However, the difference is not limited to the interpretation of and intuition about monetary policy. Focus on the nominal interest rate has also led to considerable consternation about the effectiveness of monetary policy at the zero lower bound and the possibility of indeterminate equilibria. Our money market interpretation of the NK model not only helps one think through the intuition of a monetary policy based on the path of money growth but also illustrates that concerns about equilibrium indeterminacy are overblown. Nonetheless, our model speaks to policy limitations at the zero lower bound. However, in our case, the issue is determining whether targeting the output gap or inflation is more important.

In short, this paper outlines the limitations of conventional interpretations of the NK model. Since the standard NK model is isomorphic to the money market specification, it is unclear whether the conventional interpretation or the interpretation presented here is correct. How-

7. Rich (2022) estimates a Markov-switching Bayesian VAR with quarterly US data from 1960 to 2018. His counterfactual provides an implied federal funds rate series estimate absent a regime switch away from a "scarce outside money" (outside money meaning monetary base) regime coinciding with the onset of the PRLB. That is, he estimates an implied path for the federal funds rate absent the introduction of QE-type programs as if the PRLB did not exist.

Figure 3: Filtered Observable Series, 2006 – 2019



Notes: Data and filtered observable series in quarterly percentages from estimating equations (4.1) – (4.10) with either conventional interest rate policy, equation (4.12), or the money market interpretation of the model with exogenous money supply policy, equations (4.13) and (4.14), imposing the PRLB on the interest rate, equation (4.11). The federal funds rate data is censored to missing from 2008q4 to 2015q3 in each estimation. The model dictates the length of the binding PRLB constraint spell given the notional rate dynamics. The models are filtered at the posterior means reported in Table 3. The conventional estimation exercise does not include the monetary base growth observable.

ever, considering that the money demand interpretation generates important policy implications absent from the conventional interpretation, it should give some pause to those wedded to the conventional interpretation of the model. Furthermore, model estimates under the money market interpretation match the duration of the binding PRLB constraint in the data, counter to model estimates with conventional interest rate policy. Although future research may provide alternative model specifications to generate a long PRLB period in model estimates with conventional interest rate policy, we see this as further evidence that researchers should consider our interpretation of the NK model. Contrary to the story often told, our model suggests that there might be a role for money in monetary policy analysis after all.

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Appendix

A Deriving the IS and Phillips Curves in the Textbook Model

The private economy equilibrium conditions in the textbook New Keynesian model, approximated about a zero net inflation steady state, consist of the following equations:

$$0 = \sigma (c_t - \mathbb{E}_t c_{t+1}) + i_t - \mathbb{E}_t \pi_{t+1} \tag{A.1}$$

$$w_t = \eta n_t + \sigma c_t \tag{A.2}$$

$$w_t = \text{mc}_t + a_t \tag{A.3}$$

$$\pi_t = \kappa \text{mc}_t + \beta \mathbb{E}_t \pi_{t+1} \tag{A.4}$$

$$y_t = a_t + n_t \tag{A.5}$$

$$y_t = c_t \tag{A.6}$$

where c_t is aggregate consumption, i_t is the nominal short-term interest rate, π_t is the inflation rate, w_t is the real wage, n_t is labor, mc_t is real marginal cost and acts as a time-varying labor wedge in the economy, and y_t is aggregate output. These equations, in order, are the consumption Euler equation governing the consumption-saving decision of the household, labor supply, labor demand, optimal goods price-setting, aggregate output, and the aggregate resource constraint. Variables are in terms of log-deviations from steady state, except the interest rate which is in level deviations from steady state making use of the approximation for a gross nominal interest rate, R_t, $\ln(R_t) - \ln(R) \approx (R_t - 1) - (R - 1) = R_t - R = i_t$ and the inflation rate which is simply the net inflation rate when considering the same approximation and that steady state net inflation is zero. σ is the inverse elasticity of intertemporal substitution, η is the inverse Frisch wage elasticity of labor supply, κ is the marginal cost elasticity of inflation, and β is the discount factor. a_t is a productivity shock and acts as the supply shock in the economy.

To derive the Phillips curve, consider a labor market equilibrium and substitute labor and consumption from the labor demand condition to write marginal cost in terms of aggregate output and productivity:

$$\text{mc}_t = (\sigma + \eta) y_t - (1 + \eta) a_t$$

From the optimal price-setting condition, in the flexible price case, $\kappa \to \infty$, marginal cost is fixed. Letting star super-scripts denote the flexible price economy, the flexible price level of output follows:

$$y_t^* = \frac{1 + \eta}{\eta + \sigma} a_t$$

and the labor market equilibrium condition can be written in terms of the output gap, $x_t =$

$y_t - y_t^*$, between the sticky and flexible price economies:

$$\mathrm{mc}_t = (\sigma + \eta) x_t$$

Substitute marginal cost in terms of the output gap from the optimal price-setting condition:

$$\pi_t = \kappa(\sigma + \eta) x_t + \beta \mathbb{E}_t \pi_{t+1} \tag{PC}$$

To derive the IS curve, consider the consumption Euler equation in the flexible price economy, defining the natural rate of interest as the risk-free real short-term rate in this case:

$$0 = \sigma \left(y_t^* - \mathbb{E}_t y_{t+1}^* \right) + r_t^*$$

where I have made use of the aggregate resource constraint. Subtracting this from the consumption Euler equation in the sticky price economy, again making use of the aggregate resource constraint in the sticky price case, defines the IS curve:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^* \right) \tag{IS}$$

What is the natural rate of interest? Substitute output from the consumption Euler equation in the flexible price case and assume that the productivity shock follows an AR(1) process with the AR coefficient ρ_a. The natural rate is given by:

$$r_t^* = -\sigma(1 - \rho_a) \frac{1 + \eta}{\sigma + \eta} a_t$$

B Proof of Proposition 1.

The NK model with money demand that is observationally equivalent to inflation-targeting interest rate policy absent money growth fluctuations is given by (IS), (PC), (QE), and a money supply specification. Consider the case where money growth responds to inflation and output gap variability:

$$\Delta m_t = \mu_\pi \pi_t + \mu_x x_t$$

The NK model can be written as:

$$\mathbb{E}_t \begin{bmatrix} \pi_{t+1} \\ x_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa(\sigma+\eta)}{\beta} \\ \frac{1}{\sigma}\left(\frac{1-\mu_\pi}{\varrho} - \frac{1}{\beta}\right) & 1 + \frac{1}{\sigma}\left(\frac{\kappa(\sigma+\eta)}{\beta} - \frac{\mu_x}{\varrho}\right) \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{\sigma} \end{bmatrix} r_t^*$$

which has a solution so long as both eigenvalues, λ_1 and λ_2, of $\mathbf{M}$ lie outside of the unit circle:

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0 \Rightarrow \lambda_1\lambda_2 - (\lambda_1 + \lambda_2) > -1$$

where:

$$\lambda_1\lambda_2 = \det(\mathbf{M}) = \frac{1}{\beta} + \frac{1}{\beta\sigma}\left(\frac{\kappa(\sigma+\eta)}{\beta} - \frac{\mu_x}{\varrho}\right) + \frac{\kappa(\sigma+\eta)}{\beta\sigma}\left(\frac{1-\mu_\pi}{\varrho} - \frac{1}{\beta}\right)$$
$$\lambda_1 + \lambda_2 = \operatorname{trace}(\mathbf{M}) = \frac{1}{\beta} + 1 + \frac{1}{\sigma}\left(\frac{\kappa(\sigma+\eta)}{\beta} - \frac{\mu_x}{\varrho}\right)$$

implying the determinacy condition:

$$0 < \frac{1}{\beta\sigma}\left(\frac{\kappa(\sigma+\eta)}{\beta} - \frac{\mu_x}{\varrho}\right) + \frac{\kappa(\sigma+\eta)}{\beta\sigma}\left(\frac{1-\mu_\pi}{\varrho} - \frac{1}{\beta}\right) - \frac{1}{\sigma}\left(\frac{\kappa(\sigma+\eta)}{\beta} - \frac{\mu_x}{\varrho}\right)$$

which simplifies to:

$$\mu_x < \frac{\kappa(\sigma+\eta)}{1-\beta}(1 - \varrho - \mu_\pi)$$

C Micro-founding the Non-monetary Block in the Estimated Model

The household maximizes lifetime utility:

$$\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \psi_{t+j} \left\{ \frac{\left(\frac{C_{t+j}}{A_{t+j}} - h \frac{C_{t+j-1}}{A_{t+j-1}} \left(\frac{A_{t+j-1}}{A_{t+j}} \right) \right)^{1-\sigma}}{1-\sigma} - \theta \frac{n_{t+j}^{1+\eta}}{1+\eta} \right\}$$

where the intertemporal preference shifter, ψ_t, evolves according to:

$$\psi_t = \psi_{t-1}^{\rho_\psi} \exp\left(\sigma_\psi \epsilon_{\psi,t}\right)$$

subject to:

$$C_{t+j} = W_{t+j} n_{t+j} + R_{t+j-1} \Pi_{t+j}^{-1} S_{t+j-1} + D_{t+j} - S_{t+j}$$

with first-order conditions with respect to C_t, n_t, S_t:

$$\begin{aligned}
\mu_t &= \psi_t \left(\frac{C_t}{A_t} - h \frac{C_{t-1}}{A_{t-1}} \left(\frac{A_{t-1}}{A_t} \right) \right)^{-\sigma} A_t^{-1} \\
\psi_t \theta n_t^{\eta} &= W_t \mu_t \\
\mu_t &= \beta \mathbb{E}_t \mu_{t+1} R_t \Pi_{t+1}^{-1}
\end{aligned}$$

Household equilibrium conditions where lowercase variables denote stationary variables:

$$\begin{aligned}
w_t &= \theta n_t^{\eta} \left(c_t - h \frac{c_{t-1}}{z_t} \right)^{\sigma} \\
1 &= \beta \mathbb{E}_t \frac{\psi_{t+1}}{\psi_t} \left(\frac{c_{t+1} - h \frac{c_t}{z_{t+1}}}{c_t - h \frac{c_{t-1}}{z_t}} \right)^{\sigma} \frac{R_t}{z_{t+1} \Pi_{t+1}}
\end{aligned}$$

The production sector is standard to NK models with Calvo pricing. Retailers indexed by $j \in [0,1]$ face the following firm-specific demand and use a linear production technology:

$$y_t(j) = A_t n_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} y_t$$

where productivity growth follows a stochastic process with trend growth:

$$\frac{A_t}{A_{t-1}} = z_t = z^{1-\rho_z} z_{t-1}^{\rho_z} \exp\left(\sigma_z \epsilon_{z,t}\right)$$

Cost minimization by retailers implies labor demand follows:

$$W_t = A_t \text{mc}_t$$

For price-resetting retailers, price-setting follows:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \frac{\psi_{t+s}}{\psi_t} \left(\frac{c_t - h\frac{c_{t-1}}{z_t}}{c_{t+s} - h\frac{c_{t+s-1}}{z_{t+s}}} \right)^{\sigma} A_t \left[\frac{P_t(j)}{P_{t+s}} - \mathrm{mc}_{t+s} \right] \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\varepsilon} y_{t+s}$$

With the first-order condition:

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \psi_{t+s} \left(c_{t+s} - h\frac{c_{t+s-1}}{z_{t+s}} \right)^{-\sigma} \left[(1-\varepsilon) P_{t+s}^{-1} + \varepsilon \mathrm{mc}_{t+s} P_t(j)^{-1} \right] \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\varepsilon} y_{t+s}$$

The optimal reset price, $P_t(j) = P_{\#,t}$, follows:

$$P_{\#,t} = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \psi_{t+s} \left(c_{t+s} - h\frac{c_{t+s-1}}{z_{t+s}} \right)^{-\sigma} \mathrm{mc}_{t+s} P_{t+s}^{\varepsilon} y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi)^s \psi_{t+s} \left(c_{t+s} - h\frac{c_{t+s-1}}{z_{t+s}} \right)^{-\sigma} P_{t+s}^{\varepsilon-1} y_{t+s}}$$

Written in terms of inflation, this simplifies to:

$$\begin{aligned}
\Pi_{\#,t} &= \frac{\varepsilon}{\varepsilon - 1} \Pi_t \frac{g_t}{h_t} \\
g_t &= \psi_t \left(c_t - h\frac{c_{t-1}}{z_t} \right)^{-\sigma} \mathrm{mc}_t y_t + \beta\phi \mathbb{E}_t \Pi_{t+1}^{\varepsilon} g_{t+1} \\
h_t &= \psi_t \left(c_t - h\frac{c_{t-1}}{z_t} \right)^{-\sigma} y_t + \beta\phi \mathbb{E}_t \Pi_{t+1}^{\varepsilon-1} h_{t+1}
\end{aligned}$$

Aggregating across retailers implies aggregate output, price dispersion, and the aggregate inflation rate follow:

$$\begin{aligned}
y_t &= \frac{n_t}{\Delta_t} = c_t \\
\Delta_t &= (1-\phi) \left(\frac{\Pi_t}{\Pi_{\#,t}} \right)^{\varepsilon} + \phi \Pi_t^{\varepsilon} \Delta_{t-1} \\
\Pi_t^{1-\varepsilon} &= (1-\phi) \Pi_{\#,t}^{1-\varepsilon} + \phi
\end{aligned}$$

The non-monetary block stationary equilibrium conditions can be written as:

$$\begin{aligned}
\lambda_t &= \left(y_t - h\frac{y_{t-1}}{z_t} \right)^{-\sigma} \\
w_t &= \theta \left(\Delta_t y_t \right)^{\eta} \lambda_t^{-1} \\
1 &= \beta \mathbb{E}_t \frac{\psi_{t+1}}{\psi_t} \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{z_{t+1}\Pi_{t+1}} \\
\Pi_{\#,t} &= \frac{\varepsilon}{\varepsilon - 1} \Pi_t \frac{g_t}{h_t} \\
g_t &= \psi_t \lambda_t w_t y_t + \beta\phi \mathbb{E}_t \Pi_{t+1}^{\varepsilon} g_{t+1}
\end{aligned}$$

$$h_t = \psi_t \lambda_t y_t + \beta\phi \mathbb{E}_t \Pi_{t+1}^{\varepsilon-1} h_{t+1}$$

$$\Delta_t = (1-\phi)\left(\frac{\Pi_t}{\Pi_{\#,t}}\right)^{\varepsilon} + \phi \Pi_t^{\varepsilon} \Delta_{t-1}$$

$$\Pi_t^{1-\varepsilon} = (1-\phi)\Pi_{\#,t}^{1-\varepsilon} + \phi$$

Log-linearized, $\widehat{c}_t = \ln c_t - \ln c$, around a (potentially) non-zero net inflation steady state:

$$\text{(C.1)} \qquad (z-h)\,\widehat{\lambda}_t = -\sigma\,(z\widehat{y}_t - h\widehat{y}_{t-1} + h\widehat{z}_t)$$

$$\text{(C.2)} \qquad \widehat{w}_t = \eta\left(\widehat{\Delta}_t + \widehat{y}_t\right) - \widehat{\lambda}_t$$

$$\text{(C.3)} \qquad \widehat{R}_t = (\widehat{\psi}_t - \mathbb{E}_t\widehat{\psi}_{t+1}) - \left(\mathbb{E}_t\widehat{\lambda}_{t+1} - \widehat{\lambda}_t\right) + \mathbb{E}_t\widehat{z}_{t+1} + \mathbb{E}_t\widehat{\Pi}_{t+1}$$

$$\text{(C.4)} \qquad \widehat{\Pi}_{\#,t} = \widehat{\Pi}_t + \widehat{g}_t - \widehat{h}_t$$

$$\text{(C.5)} \qquad \widehat{g}_t = (1-\beta\phi\Pi^{\varepsilon})\left(\widehat{\psi}_t + \widehat{w}_t + \widehat{y}_t + \widehat{\lambda}_t\right) + \beta\phi\Pi^{\varepsilon}\left(\varepsilon\mathbb{E}_t\widehat{\Pi}_{t+1} + \mathbb{E}_t\widehat{g}_{t+1}\right)$$

$$\text{(C.6)} \qquad \widehat{h}_t = \left(1-\beta\phi\Pi^{\varepsilon-1}\right)\left(\widehat{\psi}_t + \widehat{y}_t + \widehat{\lambda}_t\right) + \beta\phi\Pi^{\varepsilon-1}\left((\varepsilon-1)\mathbb{E}_t\widehat{\Pi}_{t+1} + \mathbb{E}_t\widehat{h}_{t+1}\right)$$

$$\text{(C.7)} \qquad \widehat{\Pi}_t = (1-\phi)\left(\frac{\Pi}{\Pi_{\#}}\right)^{\varepsilon-1}\widehat{\Pi}_{\#,t}$$

$$\text{(C.8)} \qquad \widehat{\Delta}_t = \varepsilon\widehat{\Pi}_t - \varepsilon\,(1-\phi\Pi^{\varepsilon})\,\widehat{\Pi}_{\#,t} + \phi\Pi^{\varepsilon}\widehat{\Delta}_{t-1}$$

which correspond to the equations in Section 4 where lowercase variables in the text denote log-deviations from steady state making note that: $i_t = \widehat{R}_t$, $\pi_t = \widehat{\Pi}_t$, $\pi_{\#,t} = \widehat{\Pi}_{\#,t}$, and $\delta_t = \widehat{\Delta}_t$.